

# Some properties of certain subclasses of multivalent integral operators

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## Abstract

For analytic function of the form  $f_i(z) = z^p + \sum_{n=2}^{\infty} a_n^i z^n$ , in the open unit disk, a class  $\Gamma_{\alpha}^p(C_1, C_2; \gamma)$  is introduced and some properties for  $\Gamma_{\alpha}^p(C_1, C_2; \gamma)$  of  $f_i(z)$  in relation to coefficient bounds, convex combination and convolution were obtained.

2010 Mathematics Subject Classification. **30C45**.

Keywords. Analytic, Multivalence, Coefficient bound, Convolution, Convex combination, Integral operator.

## 1 Introduction

Let  $A$  denotes the class of functions of the form  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ ,  $U = \{z \in C : |z| < 1\}$  normalized with  $f(0) = f'(0) - 1 = 0$  in the open disk,  $U = \{z \in C : |z| < 1\}$ . In [8], Seenivasagan gave a condition for the univalence of the integral operator

$$F_{\alpha, \beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \prod_{i=1}^k \left( \frac{f_i(z)}{t} \right)^{\frac{1}{\alpha}} dt \right\}^{\frac{1}{\beta}}$$

where  $f_i(z)$  is defined by

$$f_i(z) = z + \sum_{n=2}^{\infty} a_n^i z^n \tag{1.1}$$

while Makinde and Opoola in [5] obtained a condition for the starlikeness of the function

$$F_{\alpha}(z) = \int_0^z \prod_{t=1}^k \left( \frac{f_i(z)}{t} \right)^{\frac{1}{\alpha}} dt, \quad \alpha \in C \tag{1.2}$$

where  $f_i(z)$  is denoted by (1.1).

Also, Xiao-Feili et al introduced the class  $L_1^*(\beta_1, \zeta_2, \lambda)$  which is a subclass of  $A$  such that

$$L_1^*(\beta_1, \beta_2, \lambda) = \left\{ f \in A : \left| \frac{f'(z) - 1}{\beta_1 f'(z) + \beta_2} \right| \leq \lambda \right\}, 0 \leq \beta_1 \leq 1, 0 < \beta_2 \leq 1; 0 < \lambda \leq 1$$

for some  $\beta_1, \beta_2$  and some real  $\lambda$ . He further denoted  $T$  as the subclass of  $A$  consisting of the function of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0$$

and  $L^*(\beta_1, \beta_2, \lambda)$  denotes the subclass of  $L_1^*(\beta_1, \beta_2, \lambda)$  which is given by

$$L^*(\beta_1, \beta_2, \lambda) = L_1^*(\beta_1, \beta_2, \lambda) \cap T$$

for some real number  $0 \leq \beta_1 \leq 1, 0 < \beta_2 \leq 1; 0 < \lambda \leq 1$ . The class  $L^*(\beta_1, \beta_2, \lambda)$  was studied by Kim and Lee in [4], see also [1, 2, 9]. Moreover, Makinde and Oladipo in [7] introduced and studied the class  $L_\alpha(\zeta_1, \zeta_2, \lambda)$ .

Now, we defined  $f_i(z)$  by

$$f_i(z) = z^p + \sum_{n=2}^{\infty} a_n^i z^n \quad (1.3)$$

Let  $F_\alpha(z)$  be defined by (1.2), then

$$\frac{zF_\alpha''(z)}{F_\alpha'(z)} = \sum_{i=1}^k \frac{1}{\alpha} \left( \frac{zf_i'(z)}{f_i(z)} - 1 \right)$$

Let  $G(z)$  be denoted by

$$G(z) = \sum_{i=1}^k \frac{1}{\alpha} \left( \frac{zf_i'(z)}{f_i(z)} - 1 \right)$$

We define

$$\Gamma_\alpha^p(C_1, C_2, \gamma) = \left\{ f_i \in A \left| \frac{G(z) + \frac{1}{\alpha} - 1}{\zeta_1(G(z) + \frac{1}{\alpha}) + \zeta_2} \right| \leq \gamma \right\} \quad (1.4)$$

for some complex  $\zeta_1, \zeta_2, \alpha$  and for some real  $\gamma, 0 \leq |\zeta_1| \leq 1, 0 < |\zeta_2| \leq 1, |\alpha| \leq 1$  and  $0 < \gamma \leq 1$ .

## 2 Main results

**Theorem 1.** Let  $f_i(z)$  be as in (1.3) and  $F_\alpha(z)$  be as in (1.2). Then,  $f_i(z)$  is in the class  $\Gamma_\alpha^p(C_1, C_2; \gamma)$  if and only if:

$$\sum_{i=1}^n \sum_{k=p+1}^{\infty} \{k[(1 + \gamma C_1) + \alpha(\gamma C_2 - 1)]\} |a_k^i| \leq \gamma |C_1 p + \alpha C_2| - |p - \alpha|, \quad (2.1)$$

$$0 \leq C_1 \leq 1, 0 < C_2 \leq 1, 0 < \alpha \leq p.$$

*Proof.* Using equations (1.1), (1.2) we have:

$$\begin{aligned} \left| \frac{G(z) + \frac{1}{\alpha} - 1}{C_1(G(z) + \frac{1}{\alpha}) + C_2} \right| &= \left| \frac{\sum_{i=1}^n (p - \alpha) + \sum_{k=p+1}^{\infty} (k - \alpha) a_k^i z^{k-p}}{\sum_{i=1}^n (C_1 p + \alpha C_2) + \sum_{k=p+1}^{\infty} (k C_1 + \alpha C_2) a_k^i z^{k-p}} \right| \\ &\leq \frac{|p - \alpha| + \sum_{i=1}^n \sum_{k=p+1}^{\infty} (k - \alpha) |a_k^i|}{|C_1 p + \alpha C_2| - \sum_{i=1}^n \sum_{k=p+1}^{\infty} (k C_1 + \alpha C_2) |a_k^i|}. \end{aligned}$$

Let  $f_i(z)$  satisfies the inequality (2.1) then  $f_i(z) \in \Gamma_\alpha^p(C_1, C_2)$ .

Conversely, let  $f_i(z) \in \Gamma_\alpha^p(C_1, C_2)$  then

$$\sum_{i=1}^n \sum_{k=p+1}^{\infty} \{k[(1 + \gamma C_1) + \alpha(\gamma C_2 - 1)]\} |a_k^i| \leq |C_1 p + \alpha C_2| - |p - \alpha|.$$

Q.E.D.

**Corollary 1.** If  $f_i(z) \in \Gamma_\alpha^p(C_1, C_2; \gamma)$ , we have:

$$\sum_{i=1}^n \sum_{k=p+1}^{\infty} |a_k^i| \leq \frac{|C_1 p + \alpha C_2| - |p - \alpha|}{k[(1 + \gamma C_1) + \alpha(\gamma C_2 - 1)]}.$$

**Theorem 2.** Let  $f_i(z) \in \Gamma_\alpha^p(C_1, C_2)$  and  $g_i(z)$  defined by  $g_i(z) = z^p + \sum_{k=p+1}^{\infty} b_k^i z^k$  be in the same  $\Gamma_\alpha^p(C_1, C_2)$ . Then the function  $h_i(z)$  defined by

$$h_i(z) = (1 - \lambda)f_i(z) + \lambda g_i(z) = z^p + \sum_{k=p+1}^{\infty} C_k^i z^k$$

is also in the class  $\Gamma_\alpha^p(C_1, C_2)$  where

$$C_k^i = (1 - \lambda)a_k^i + \lambda b_k^i, \quad 0 \leq \lambda \leq 1.$$

*Proof.* Suppose each of  $f_i(z)$  and  $g_i(z)$  in the  $\Gamma_\alpha^p(C_1, C_2; \lambda)$ . Then we have

$$\begin{aligned} \sum_{i=1}^n \sum_{k=p+1}^{\infty} \{k[(1 + \gamma C_1) + \alpha(\gamma C_2 - 1)]\} |C_k^i| &= \sum_{i=1}^n \sum_{k=p+1}^{\infty} \{k[(1 + \gamma C_1) + \alpha(\gamma C_2 - 1)]\} |(1 - \lambda)a_k^i + \lambda b_k^i| \\ &= (1 - \lambda) \sum_{i=1}^n \sum_{k=p+1}^{\infty} \{k[(1 + \gamma C_1) + \alpha(\gamma C_2 - 1)]\} |a_k^i| \\ &\quad + \lambda \sum_{i=1}^n \sum_{k=p+1}^{\infty} \{k[(1 + \gamma C_1) + \alpha(\gamma C_2 - 1)]\} |b_k^i| \\ &\leq (1 - \lambda)(|C_1 p + \alpha C_2| - |p - \alpha|) \\ &\quad + \lambda(|C_1 p + \alpha C_2| - |p - \alpha|) \\ &= |C_1 p + \alpha C_2| - |p - \alpha| \end{aligned}$$

which shows that the convex combination of  $f_i(z)$  and  $g_i(z)$  is in the class  $\Gamma_\alpha^p(C_1, C_2; \gamma)$ . Q.E.D.

**Theorem 3.** Let  $f_i(z)$  be as in (3) and  $F_\alpha(z)$  be as in (1.2) then the function  $C_i(z)$  defined by

$$C_i(z) = z^p + \sum_{k=p+1}^{\infty} a_k^i b_k^i z^k$$

is in the class  $\Gamma_\alpha^p(C_1, C_2; \gamma)$  if and only if

$$\sum_{i=1}^n \sum_{k=p+1}^{\infty} \{k[(1 + \gamma C_1) + \alpha(\gamma C_2 - 1)]\} |a_k^i b_k^i| \leq \gamma |C_1 p + \alpha C_2| - |p - \alpha|,$$

$$0 \leq C_1 \leq 1, 0 < C_1 \leq 1, 0 < \alpha \leq p.$$

*Proof.* Following the procedure of the proof of the Theorem 1, we obtain the result. Q.E.D.

**Theorem 4.** Let the function  $\psi_i(z)$  be in the class  $\Gamma_\alpha^p(C_1, C_2; \gamma)$  and the function  $\zeta_i(z)$  defined by

$$\zeta_i(z) = z^p + \sum_{k=p+1}^{\infty} A_k^i B_k^i z^k$$

be in the same  $\Gamma_\alpha^p(C_1, C_2; \gamma)$ . Then the function  $H(z)$  defined by

$$H(z) = (1 - \lambda)\zeta_i(z) + \lambda\psi_i(z) = z^p + \sum_{k=p+1}^{\infty} C_k^i$$

is also in the class  $\Gamma_\alpha^p(C_1, C_2; \gamma)$ , where

$$C_k^i = (1 - \lambda)a_k^i b_k^i + \lambda A_k^i B_k^i, \quad 0 \leq \lambda \leq 1.$$

*Proof.* The proof of the theorem is similar to that of Theorem 2, thus, we omit the proof. Q.E.D.

**Corollary 2.** Let  $f_i(z)$  be as in (1.3) and  $F_\alpha$  be as in (1.4). Then  $f_i(z)$  is in the class  $\Gamma_\alpha^p(C_1, C_2; \gamma)$  if and only if

$$\sum_{i=1}^n \sum_{k=p+1}^{\infty} \{k[(1 + \gamma C_1) + \alpha(\gamma C_2 - 1)]\} |a_k^i| \leq \gamma |C_1 p + \alpha C_2| - |p - \alpha|,$$

$$0 \leq C_1 \leq 1, 0 < C_1 \leq 1, 0 < \alpha \leq p.$$

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